Errata: Proof of Lemma C.2

There is a major flaw in the proof of this lemma. At the bottom of this page, it is argued that Theorem A.116 proves that the probability that a binomially distributed variable with mean of at least ε exceeds a value of $\frac{m\varepsilon}{2}$ is at least $\frac{1}{2}$. This is wrong. In order to prove this statement we need a different theorem. In the official errata we give a short proof of this statement using Theorem A.110 which, unfortunately, makes the condition $m\varepsilon > 2$ slightly worse ($m\varepsilon > 8$). Here, we give a longer proof due to Mingrui Wu which shows the desired result from first principles.

Theorem 0.1 (Binomial mean deviation bound) *Let* $\mathsf{X}_1, \ldots, \mathsf{X}_n$ *be independent random variables such that, for all* $i \in \{1, \ldots, n\}$, $\mathbf{P}_{\mathsf{X}_i}(\mathsf{X}_i = 1) = 1 - \mathbf{P}_{\mathsf{X}_i}(\mathsf{X}_i = 0) = \mathbf{E}_{\mathsf{X}_i}\left[\mathsf{X}_i\right] = \mu$. *Then, for all* $\varepsilon \in \left(\frac{2}{n}, \mu\right)$ *we have*

$$\mathbf{P}_{\mathsf{X}^n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathsf{X}_i \geq \frac{\varepsilon}{2}\right) > \frac{1}{2}\,.$$

Proof Since $\mu > \varepsilon$ it suffices to show

$$\mathbf{P}_{\mathsf{X}^n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathsf{X}_i \geq \frac{\varepsilon}{2}\right) \geq \mathbf{P}_{\mathsf{X}^n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathsf{X}_i \geq \frac{\mu}{2}\right) > \frac{1}{2}\,,$$

assuming that $n\mu > 2$. This statement is equivalent to

$$\mathbf{P}_{\mathsf{X}^n}\left(\sum_{i=1}^{n}\mathsf{X}_i < \frac{n\mu}{2}\right) \leq \frac{1}{2}\,. \tag{1}$$

Let l be the largest integer such that $l < \frac{n\mu}{2}$. Since $\mu \in [0, 1]$ and n is an integer we know that $2l + 1 \leq n$. Note that $\mathsf{S} := \sum_{i=1}^{n}\mathsf{X}_i$ is binomially distributed with parameters n and μ (see Table A.2). Thus, (1) is equivalent to

$$\sum_{j=0}^{l}\binom{n}{j}\mu^{j}(1-\mu)^{n-j} \leq \frac{1}{2}\,. \tag{2}$$

Case 1: $\mu > \frac{1}{2}$ In this case $\mu > 1-\mu$ and for $j \in \{0,\ldots,l\}$ we have $j < n-j$ so it follows that

$$\binom{n}{j}\mu^j (1-\mu)^{n-j} < \binom{n}{j}\mu^{n-j}(1-\mu)^j = \binom{n}{n-j}\mu^{n-j}(1-\mu)^j \ .$$

Hence, double summation of (2) gives

$$\begin{aligned} 2\sum_{j=0}^{l}\binom{n}{j}\mu^j(1-\mu)^{n-j} &< \sum_{j=0}^{l}\binom{n}{j}\mu^j(1-\mu)^{n-j} + \sum_{j=n-l}^{n}\binom{n}{j}\mu^j(1-\mu)^{n-j} \\ &\leq \sum_{j=0}^{n}\binom{n}{j}\mu^j(1-\mu)^{n-j} = 1\,,. \end{aligned}$$

Case 2: $\mu \leq \frac{1}{2}$ By assumption $n\mu > 2$ and thus $l \leq \frac{n}{4}$ and $n > 4$. In the rest of the proof we will show that

$$\forall j \in \{1,\ldots,l\}: \quad \binom{n}{j}\mu^j(1-\mu)^{n-j} < \binom{n}{j+l}\mu^{j+l}(1-\mu)^{n-j-l}\,, \tag{3}$$

$$(1-\mu)^n < \binom{n}{2l+1}\mu^{2l+1}(1-\mu)^{n-2l-1}\,. \tag{4}$$

Using these two results, (2) can be seen to hold by noticing that (3) and (4) imply

$$\begin{aligned} \sum_{j=0}^{l}\binom{n}{j}\mu^j(1-\mu)^{n-j} &= \sum_{j=1}^{l}\binom{n}{j}\mu^j(1-\mu)^{n-j} + (1-\mu)^n \\ &< \sum_{j=l+1}^{2l+1}\binom{n}{j}\mu^j(1-\mu)^{n-j}\ . \end{aligned}$$

Hence, double summation of (2) again gives

$$\begin{aligned} 2\sum_{j=0}^{l}\binom{n}{j}\mu^j(1-\mu)^{n-j} &< \sum_{j=0}^{2l+1}\binom{n}{j}\mu^j(1-\mu)^{n-j} \\ &\leq \sum_{j=0}^{n}\binom{n}{j}\mu^j(1-\mu)^{n-j} = 1\,, \end{aligned}$$

where we used the fact that $2l+1 \leq n$. It remains to show (3) and (4). In order to prove (3) we divide the right hand side by the left hand side. For the jth term this results in

$$
\begin{aligned}
\frac{\binom{n}{j+l}\mu^{j+l}\,(1-\mu)^{n-j-l}}{\binom{n}{j}\mu^{j}\,(1-\mu)^{n-j}} &= \prod_{t=1}^{l}\frac{\mu}{1-\mu}\cdot\frac{n-j-l+t}{j+t}\\
&\geq \prod_{t=1}^{l}\frac{\mu}{1-\mu}\cdot\frac{n-2l+t}{l+t}\\
&= \prod_{t=1}^{l}\frac{\mu}{1-\mu}\left(1+\frac{n-3l}{l+t}\right)\\
&\geq \prod_{t=1}^{l}\frac{\mu}{1-\mu}\cdot\frac{n-l}{2l}\\
&= \left(\frac{\mu}{1-\mu}\cdot\frac{n-l}{2l}\right)^{l}\\
&> \left(\frac{\mu}{1-\mu}.\frac{n-\frac{n\mu}{2}}{n\mu}\right)^{l}\\
&= \left(\frac{1-\frac{\mu}{2}}{1-\mu}\right)^{l} > 1\,,
\end{aligned}
$$

where we used $j \leq l$ in the second line, $t \leq l$ and $n-3l \geq 0$ in the third line and $l < \frac{n\mu}{2}$ in the penultimate line. In order to show (4) we assume $l \geq 1$; otherwise the statement follows easily. Again, dividing the right hand side of (4) by the left hand side of (4) we obtain

$$
\begin{aligned}
&\frac{\binom{n}{2l+1}\mu^{2l+1}\,(1-\mu)^{n-2l-1}}{(1-\mu)^{n}}\\
&\quad= \prod_{t=1}^{2l+1}\frac{\mu}{1-\mu}\cdot\frac{n-2l-1+t}{t}\\
&\quad= \left(\prod_{t=2}^{2l}\frac{\mu}{1-\mu}\cdot\frac{n-2l-1+t}{t}\right)\left(\frac{n\,(n-2l)}{2l+1}\left(\frac{\mu}{1-\mu}\right)^{2}\right)\\
&\quad> \left(\prod_{t=2}^{2l}\frac{\mu}{1-\mu}\cdot\frac{n-1}{2l}\right)\left(\frac{n\,(n-n\mu)}{2l+1}\left(\frac{\mu}{1-\mu}\right)^{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{n\mu - \mu}{2l - 2l\mu}\right)^{2l-1} \left(\frac{n^2\mu^2}{(2l+1)\,(1-\mu)}\right) \\
&> \left(\frac{2l - \mu}{2l - 2l\mu}\right)^{2l-1} \left(\frac{n^2\mu^2}{2l+1}\right) \\
&> \left(\frac{2l - \mu}{2l - 2l\mu}\right)^{2l-1} \left(\frac{n^2\mu^2}{n\mu+1}\right) > 1\,,
\end{aligned}
$$

where the third and fifth line uses $t \leq 2l < n\mu$ and the last line uses $n\mu > 2$.

The theorem is proven. ■